

SOME HOMOLOGICAL CRITERIA FOR REGULAR, COMPLETE INTERSECTION AND GORENSTEIN RINGS

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ABSTRACT. In 1969, Kunz proved that a noetherian local ring of positive characteristic is regular if and only if its Frobenius homomorphism is flat. Since then, a number of criteria for regularity, complete intersection, Gorenstein and Cohen-Macaulay rings in terms of homological conditions of the Frobenius homomorphism have been obtained by other authors. We show here how a result of Avramov can play an important part in these criteria.

In [15], the following theorem is proved:

Theorem (Kunz) Let A be a noetherian local ring containing a field of characteristic $p > 0$, let $\phi : A \rightarrow A, \phi(a) = a^p$ be the Frobenius homomorphism, and let ϕA be the ring A considered as A -module via ϕ . The following are equivalent:

- (i) A is regular
- (ii) ϕA is a flat A -module.

In a different context, we have [3]:

Theorem (Avramov) Let $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ be a local homomorphism of noetherian local rings. Let a_1, \dots, a_r be a minimal set of generators of the maximal ideal \mathfrak{m} of A , and $f(a_1), \dots, f(a_r), b_1, \dots, b_s$ a set of generators of the ideal \mathfrak{n} . Assume that $fd_A(B) < \infty$ (that is, $Tor_n^A(B, -) = 0$ for all $n \gg 0$). Then, the induced homomorphism between the first Koszul homology modules

$$H_1(a_1, \dots, a_r; A) \otimes_k l \rightarrow H_1(f(a_1), \dots, f(a_r), b_1, \dots, b_s; B)$$

is injective.

In order to see the strength of this theorem, let us mention only two particular cases:

- When f is flat, the theorem was previously proved also by Avramov [2] and allowed him to prove that the complete intersection property localizes.
- When $B = k$ is the residue field of A , the theorem gives the well known Serre's characterization of regularity by finiteness of the global dimension.

In [18], Kunz theorem is improved as follows: if $fd_A(\phi A) < \infty$ then A is regular, and in [11] similar characterizations of complete intersection rings in terms of homological dimensions of the Frobenius homomorphisms were obtained. Both results used is an essential way in their proofs the theorem of Avramov cited above.

Using different methods, more criteria of this kind were found for the Gorenstein and Cohen-Macaulay properties ([13], [20], [14]) and also some results were extended in [20], [10], [17], [9] from the particular case of the Frobenius endomorphism to

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the more general case of a contracting endomorphism $f : (A, \mathfrak{m}, k) \rightarrow (A, \mathfrak{m}, k)$, that is, a homomorphism satisfying $f^i(\mathfrak{m}) \subset \mathfrak{m}^2$ for some $i > 0$ (and in particular containing a field [9, 5.9]).

In this paper, we show that some of those results can also be deduced from the theorem of Avramov. Moreover, this way allows to improve them, by extending them to a class of homomorphisms larger than the contracting ones. In particular, we can produce examples of homomorphisms (not necessarily endomorphisms) of rings that do not contain a field.

For convenience of the reader, we will start by recalling here some facts of André-Quillen homology that we are going to use through the paper. Associated to a homomorphism of (always commutative) rings $f : A \rightarrow B$ and to a B -module M we have André-Quillen homology B -modules $H_n(A, B, M)$ for all integers $n \geq 0$, which are functorial in all three variables.

1. If $B = A/I$, then $H_0(A, B, M) = 0$, $H_1(A, B, M) = I/I^2 \otimes_B M$ [1, 4.60, 6.1].
2. (Base change) Let $A \rightarrow B$, $A \rightarrow C$ be ring homomorphisms such that B or C is flat as A -module, and let M be a $B \otimes_A C$ -module. Then $H_n(A, B, M) = H_n(C, B \otimes_A C, M)$ for all n [1, 4.54].
3. Let B be an A -algebra, C a B -algebra and M a flat C -module. Then $H_n(A, B, M) = H_n(A, B, C) \otimes_C M$ for all n [1, 3.20].
4. (Jacobi-Zariski exact sequence) If $A \rightarrow B \rightarrow C$ are ring homomorphisms and M is a C -module, we have a natural exact sequence [1, 5.1]

$$\begin{array}{ccccccc} & & & & \dots & \rightarrow & H_{n+1}(B, C, M) \rightarrow \\ & & & & H_n(A, B, M) \rightarrow & H_n(A, C, M) \rightarrow & H_n(B, C, M) \rightarrow \\ & & & & H_{n-1}(A, B, M) \rightarrow & \dots & \rightarrow H_0(B, C, M) \rightarrow 0 \end{array}$$

5. If $K \rightarrow L$ is a field extension and M an L -module, we have $H_n(K, L, M) = 0$ for all $n \geq 2$ [1, 7.4]. So if $A \rightarrow K \rightarrow L$ are ring homomorphisms with K and L fields, from 4 we obtain $H_n(A, K, L) = H_n(A, L, L)$ for all $n \geq 2$, which, using 3, gives $H_n(A, K, K) \otimes_K L = H_n(A, L, L)$ for all $n \geq 2$.

6. If I is an ideal of a noetherian local ring (A, \mathfrak{m}, k) , then the following are equivalent:

- (i) I is generated by a regular sequence
- (ii) $H_2(A, A/I, k) = 0$
- (iii) $H_n(A, A/I, M) = 0$ for any A/I -module M for all $n \geq 2$ [1, 6.25].

In particular, a noetherian local ring (A, \mathfrak{m}, k) is regular if and only if $H_2(A, k, k) = 0$.

7. If (A, \mathfrak{m}, k) is a noetherian local ring and \hat{A} is its \mathfrak{m} -completion, then $H_n(A, k, k) = H_n(\hat{A}, k, k)$ for all $n \geq 0$ [1, 10.18].

8. If $(A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ is a local homomorphism of noetherian local rings with $fd_A(B) < \infty$, then the theorem of Avramov cited above says that the homomorphism $H_2(A, l, l) \rightarrow H_2(B, l, l)$ is injective [1, 15.12] (see details in the proof of [16, 4.2.2]).

Definition 1. Let $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ be a local homomorphism of noetherian local rings. We say that f has the h_2 -vanishing property if the homomorphism induced by f

$$H_2(A, l, l) \rightarrow H_2(B, l, l)$$

vanishes.

Proposition 2. If $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ has the h_2 -vanishing property and there exists a local homomorphism of noetherian local rings $B \rightarrow C$ such that $fd_A(C) < \infty$, then A is a regular local ring.

Proof. Since the composition $A \rightarrow B \rightarrow C$ has the h_2 -vanishing property, we can assume $B=C$. Therefore, the zero map $H_2(A, l, l) \rightarrow H_2(B, l, l)$ is injective by Avramov's theorem, so $H_2(A, l, l) = 0$ and then A is regular. \square

Examples 3. (i) If the homomorphism $f : A \rightarrow B$ factorizes into local homomorphisms $f : A \rightarrow R \rightarrow B$ where R is a regular local ring, then f has the h_2 -vanishing property.

(ii) Let $f : (A, \mathfrak{m}, k) \rightarrow (A, \mathfrak{m}, k)$ be a contracting homomorphism [10, §12], that is, there exists some $i > 0$ such that $f^i(\mathfrak{m}) \subset \mathfrak{m}^2$. That means that for any $t > 0$, $f^j(\mathfrak{m}) \subset \mathfrak{m}^t$ for some $j > 0$. We will see that some power of f has the h_2 -vanishing property and how to apply Proposition 2 to these endomorphisms.

More precisely, assume that $f : A \rightarrow A$ is contracting and there exists a noetherian A -algebra C which has some prime ideal that contracts in A to \mathfrak{m} such that $fd_A(f^i C) < \infty$ for some i . Then we will prove that A is regular. This case was proved in [20, Remark 4.4] and includes the particular case of a finite A -algebra C with $fd_A(f^i C) < \infty$, which was proved with different methods in [9] in the more general case of a finite A -module $C \neq 0$ with $fd_A(f^i C) < \infty$.

Assume first that A is complete. Since A is equicharacteristic [9, 5.9] (the subring of elements fixed by f is a field), A has a coefficient field k_0 , and there exists a surjective ring homomorphism $R := k_0[[X_1, \dots, X_n]] \rightarrow A$ sending X_1, \dots, X_n to a set of generators x_1, \dots, x_n of the maximal ideal \mathfrak{m} of A . Let I be its kernel. Define a ring endomorphism g of R as follows: for each j , choose a power series $P_j(x_1, \dots, x_n)$ in x_1, \dots, x_n over k_0 representing $f^i(x_j)$; then define $g|_{k_0} := f^i|_{k_0}$, $g(X_j) := P_j(X_1, \dots, X_n)$. So defined, g is a lifting of f^i , and in particular $g(I) \subset I$. Moreover, since $f^i(\mathfrak{m}) \subset \mathfrak{m}^2$, avoiding unnecessary terms, we can assume that each P_j has order ≥ 2 and so $g(\mathfrak{q}) \subset \mathfrak{q}^2$, where $\mathfrak{q} = (X_1, \dots, X_n)$ is the maximal ideal of R .

Therefore, for any $t > 0$ there exists some s such that $g^s(\mathfrak{q}) \subset \mathfrak{q}^t$, and then $g^s(I) \subset \mathfrak{q}^t \cap I$. By the Artin-Rees lemma, some s verifies $g^s(I) \subset \mathfrak{q}I$. Consider the

Jacobi-Zariski exact sequence associated to $R \rightarrow A \rightarrow k$

$$0 = H_2(R, k, k) \rightarrow H_2(A, k, k) \rightarrow I/\mathfrak{q}I$$

The homomorphism induced by f^{is} on $H_2(A, k, k)$ is induced by g^s on $I/\mathfrak{q}I$ which is zero. We have proved that if $f : A \rightarrow A$ is contracting, then some power f^{is} of f has the h_2 -vanishing property when A is complete. But since $H_2(A, k, k) = H_2(\hat{A}, k, k)$, this is also valid when A is not complete.

Finally, let $f : A \rightarrow A$ be contracting and assume that there exists a noetherian local A -algebra (C, \mathfrak{p}, l) such that $fd_A(f^i C) < \infty$ for some i . By Avramov's theorem, the homomorphism $H_2(A, k, l) \rightarrow H_2(C, l, l)$ induced by $A \xrightarrow{f^i} A \rightarrow C$ is injective, and then so is the endomorphism $H_2(A, k, k) \rightarrow H_2(A, k, k)$ induced by f^i . Thus the homomorphism endomorphism $H_2(A, k, k) \rightarrow H_2(A, k, k)$ induced by f^{is} is also injective. But f^{is} has the h_2 -vanishing property, so $H_2(A, k, k) = 0$ and then A is regular.

(iii) Those two sources of examples (i) and (ii) can be combined (via tensor product) in some cases, using [1, 5.21] and standard properties of André-Quillen homology to check the h_2 -vanishing property.

Remark 4. It can be proved that if $f : A \rightarrow A$ is contracting, then for any $n \geq 0$, there exists s (depending on n) such that f^s has the h_n -vanishing property, that is, it induces the zero map $H_n(A, k, k) \rightarrow H_n(A, k, k)$.

Avramov's theorem also allows us to obtain similar criteria for complete intersection, Gorenstein, and Cohen-Macaulay rings, provided we use the adequate definitions for homological dimensions in terms of "deformations" (see [5, §8]) and flat dimension. We start by recalling the definition of upper complete intersection dimension introduced in [19] (see also [8]).

We say that a finite module $M \neq 0$ over a noetherian local ring A has finite upper complete intersection dimension and denote it by $\text{CI}^*\text{-dim}_A(M) < \infty$ if there exists a flat local homomorphism of noetherian local rings $(A, \mathfrak{m}, k) \rightarrow (A', \mathfrak{m}', k')$ such that $A' \otimes_A k$ is a regular local ring, and a surjective homomorphism of noetherian local rings $Q \rightarrow A'$ with kernel generated by a regular sequence, such that $pd_Q(M \otimes_A A') < \infty$, where pd denotes projective dimension.

If $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ is a local homomorphism of noetherian local rings, a Cohen factorization of f is a factorization $A \xrightarrow{i} R \xrightarrow{p} B$ of f where R is a noetherian local ring, i is a flat local homomorphism, $R \otimes_A k$ is a regular local ring and p is surjective. If B is complete, a Cohen factorization always exists [7].

We say that a local homomorphism of noetherian local rings $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ is of finite upper complete intersection dimension (and denote it by $\text{CI}^*\text{-dim}(f) < \infty$) if there exists a Cohen factorization $A \rightarrow R \rightarrow \hat{B}$, such that $\text{CI}^*\text{-dim}_R(\hat{B}) < \infty$.

Lemma 5. (Essentially [4, lemma 1.7]) *Let $(A, k) \rightarrow (R, l) \rightarrow (D, E)$ be local homomorphisms of noetherian local rings such that R is a flat A -module and $R \otimes_A k$ is regular. Then $H_n(A, D, E) = H_n(R, D, E)$ for all $n \geq 2$.*

Proof. By flat base change $H_n(A, R, E) = H_n(k, R \otimes_A k, E)$, and by the Jacobi-Zariski exact sequence associated to $k \rightarrow R \otimes_A k \rightarrow E$ we have $H_n(k, R \otimes_A k, E) =$

$H_{n+1}(R \otimes_A k, E, E) = 0$ for all $n \geq 2$. So the Jacobi-Zariski exact sequence

$$\dots \rightarrow H_n(A, R, E) \rightarrow H_n(A, D, E) \rightarrow H_n(R, D, E) \rightarrow H_{n-1}(A, R, E) \rightarrow \dots$$

gives isomorphisms $H_n(A, D, E) = H_n(R, D, E)$ for all $n \geq 3$ and an exact sequence

$$0 \rightarrow H_2(A, D, E) \rightarrow H_2(R, D, E) \rightarrow H_1(A, R, E) \xrightarrow{\alpha} H_1(A, D, E) \rightarrow \dots$$

The injectivity of α follows from the commutative diagram with exact upper row

$$\begin{array}{ccccc} 0 = H_2(R \otimes_A k, E, E) & \longrightarrow & H_1(k, R \otimes_A k, E) & \longrightarrow & H_1(k, E, E) \\ & & \uparrow \simeq & & \uparrow \\ & & H_1(A, R, E) & \xrightarrow{\alpha} & H_2(A, D, E) \end{array}$$

□

Proposition 6. *If $f : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ has the h_2 -vanishing property and there exists a local homomorphism of noetherian local rings $g : (B, \mathfrak{n}, l) \rightarrow (C, \mathfrak{p}, E)$ such that $CI^*\text{-dim}(gf) < \infty$, then A is a complete intersection ring.*

Proof. Let $A \rightarrow R \rightarrow \hat{C}$ be a Cohen factorization, $R \rightarrow R'$ a flat local homomorphism with regular closed fiber, $Q \rightarrow R'$ a surjective homomorphism of noetherian local rings with kernel generated by a regular sequence such that $pd_Q(\hat{C} \otimes_R R') < \infty$. We have a commutative triangle

$$\begin{array}{ccc} & H_2(R, E, E) & \\ \beta \nearrow & & \searrow \gamma \\ H_2(A, E, E) & \xrightarrow{\alpha} & H_2(\hat{C}, E, E) \end{array}$$

where $\alpha = 0$ since f has the h_2 -vanishing property, and β is surjective by Lemma 5. Then $\gamma = 0$ and so the homomorphism $H_2(R, E', E') \rightarrow H_2(\hat{C}, E', E')$ also vanishes, where E' is the residue field of R' and $\hat{C} \otimes_R R'$. We have a commutative diagram

$$\begin{array}{ccc} H_2(R, E', E') & \xrightarrow{0} & H_2(\hat{C}, E', E') \\ \downarrow \lambda & & \downarrow \\ H_2(R', E', E') & \xrightarrow{\mu} & H_2(\hat{C} \otimes_R R', E', E') \end{array}$$

where λ is an isomorphism by Lemma 5. We see that $\mu = 0$, that is $R' \rightarrow \hat{C} \otimes_R R'$ has the h_2 -vanishing property. Composing with $Q \rightarrow R'$, we deduce that $Q \rightarrow \hat{C} \otimes_R R'$ has the h_2 -vanishing property. By Proposition 2, Q is a regular local ring, and then R' is a complete intersection ring. By flat descent ([2] or [3]), A is a complete intersection ring. □

Following in part [21] we define the upper Gorenstein dimension as follows. If $M \neq 0$ is a finite module over a noetherian local ring A , we say that M has finite upper Gorenstein dimension if there exists a flat local homomorphism of noetherian local rings $A \rightarrow A'$ with regular closed fiber and a surjective homomorphism of noetherian local rings $Q \rightarrow A'$ verifying that $Ext_Q^n(A', Q) = A'$ for some n and $Ext_Q^i(A', Q) = 0$ for all $i \neq n$, such that $pd_Q(M \otimes_A A') < \infty$. We say that a local homomorphism of noetherian local rings $A \rightarrow B$ has finite upper Gorenstein

dimension if for some Cohen factorization $A \rightarrow R \rightarrow \hat{B}$, the R -module \hat{B} has finite upper Gorenstein dimension.

Proposition 7. *If $A \rightarrow B$ has the h_2 -vanishing property and there exists a local homomorphism of noetherian local rings $g : B \rightarrow C$ such that gf has finite upper Gorenstein dimension, then A is a Gorenstein ring.*

Proof. As in the proof of Proposition 6, we know that there exist a Cohen factorization $A \rightarrow R \rightarrow \hat{C}$, a flat local homomorphism $R \rightarrow R'$, a regular local ring Q and a surjective homomorphism $Q \rightarrow R'$ such that $\text{Ext}_Q^n(R', Q) = R'$ for some n and $\text{Ext}_Q^i(R', Q) = 0$ for all $i \neq n$. Now if E' is the residue field of R' , from the change of rings spectral sequence

$$E_2^{pq} = \text{Ext}_{R'}^p(E', \text{Ext}_Q^q(R', Q)) \Rightarrow \text{Ext}_Q^{p+q}(E', Q)$$

we deduce that R' is Gorenstein. By flat descent, A is also Gorenstein. \square

Remark 8. For a noetherian local ring S let $\text{cmd}(S) := \dim(S) - \text{depth}(S)$ [12, 0_{IV}16.4.9], [6]. If we say that a finite A -module $M \neq 0$ has finite Cohen-Macaulay dimension when there exists a flat local homomorphism of noetherian local rings $A \rightarrow A'$ with regular closed fiber and a surjective homomorphism of noetherian local rings $Q \rightarrow A'$ verifying that $\text{cmd}(Q) = \text{cmd}(A')$ (compare with [6, (3.2)]) and $\text{pd}_Q(M \otimes_A A') < \infty$, then we have a similar criterion for Cohen-Macaulay rings.

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